

# Projective Ring Lines and Their Generalisations

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## Abstract

We give a survey on projective ring lines and some of their substructures which in turn are more general than a projective line over a ring.

**Keywords:** Projective line over a ring, distant graph, connected component, elementary linear group, subspace of a chain geometry, Jordan system, projective line over a strong Jordan system

## 1 Distant graph and connected components

The *projective line*  $\mathbb{P}(R)$  over any ring  $R$  (associative with  $1 \neq 0$ ) can be defined in terms of the free left  $R$ -module  $R^2$  as follows [11], [24]: It is the orbit of a starter point  $R(1, 0)$  under the action of the general linear group  $\mathrm{GL}_2(R)$  on  $R^2$ . A basic notion on  $\mathbb{P}(R)$  is its *distant relation*: Two points are called distant (in symbols:  $\Delta$ ) if they can be represented by the elements of a two-element basis of  $R^2$ . The *distant graph*  $(\mathbb{P}(R), \Delta)$  has as vertices the points of  $\mathbb{P}(R)$  and as edges the pairs of distant points. The distant graph is connected precisely when  $\mathrm{GL}_2(R)$  is generated by the *elementary linear group*  $E_2(R)$ , i.e., the subgroup of  $\mathrm{GL}_2(R)$  which is generated by elementary transvections, together with the set of all invertible diagonal matrices [7]. The orbit of  $R(1, 0)$  under  $E_2(R)$  is a connected component of the distant graph. It admits a parametrisation in terms of infinitely many formulas [7], [8]. The situation is less intricate for a ring  $R$  of *stable rank* 2 (see

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[15], [36], or [37]), as it gives rise to a connected distant graph with diameter  $\leq 2$ . The above-mentioned parametrisation turns into *Bartolone's parametrisation* [1] of  $\mathbb{P}(R)$ , namely

$$\mathbb{P}(R) = \{R(t_2 t_1 - 1, t_2) \mid t_1, t_2 \in R\} \quad (R \text{ of stable rank } 2).$$

Refer to the seminal paper of P. M. Cohn [16] for the algebraic background, and to the work of A. Blunck [5], [6] for orbits of the point  $R(1, 0)$  under other subgroups of  $\mathrm{GL}_2(R)$ .

## 2 Chain Geometries, subspaces and Jordan Systems

Let  $R$  be an algebra over a commutative field  $K$ ; by identifying  $K$  with  $K \cdot 1_R$  the projective line  $\mathbb{P}(K)$  is embedded in  $\mathbb{P}(R)$ . For  $R \neq K$  the projective line  $\mathbb{P}(R)$  can be considered as the point set of the *chain geometry*  $\Sigma(K, R)$ ; the  $\mathrm{GL}_2(R)$  orbit of  $\mathbb{P}(K)$  is the set of *chains* [11], [24]. The geometries of Möbius, Minkowski and Laguerre are well known examples of chain geometries [2]. A crucial property is that any three mutually distinct points are on a unique chain. The chain geometry  $\Sigma(K, R)$  may be viewed as a refinement of the distant graph, since two points of  $\mathbb{P}(R)$  are distant if, and only if, they are on a common chain. There are cases though, when the word “refinement” is inappropriate in its strict sense: Let  $R = \mathrm{End}_F(V)$  be the endomorphism ring of a vector space  $V$  over a (not necessarily commutative) field  $F$  and let  $K$  denote the *centre* of  $F$ . Then the  $K$ -chains of  $\mathbb{P}(R)$  can be defined solely in terms of the distant graph  $(\mathbb{P}(R), \Delta)$  [10].

Each chain geometry  $\Sigma(K, R)$  is a *chain space*; see [11], where also the precise definition of *subspaces* of a chain space is given. The algebraic description of subspaces of  $\Sigma(K, R)$  is due to A. Herzer [23] and H.-J. Kroll [29], [30], [31]. It is based on the following notions: A *Jordan system* is a  $K$ -subspace of  $R$  satisfying two extra conditions: (i)  $1 \in J$ ; (ii) If  $b \in J$  has an inverse in  $R$  then  $b^{-1} \in J$ . (See [33] for relations with *Jordan algebras* and *Jordan pairs* and compare with [18], [34].) A Jordan system  $J$  is called *strong* if it satisfies a (somewhat technical) condition which guarantees the existence of “many” invertible elements in  $J$ . Strong Jordan systems are closed under *triple multiplication*, i. e.,  $xyx \in J$  for all  $x, y \in J$ . The *projective line*  $\mathbb{P}(J)$  over a strong Jordan system  $J \subset R$  is defined by restricting the *parameters*  $t_1, t_2$  to  $J$  in Bartolone's parametrisation. We wish to emphasise that in general a point of  $\mathbb{P}(J)$  cannot be written as  $R(a, b)$  with  $a, b \in J$ , unless  $J$  is even a subalgebra of  $R$ . The main theorem about subspaces is as follows: If  $R$  is a strong algebra then any connected subspace of  $\Sigma(K, R)$  is projectively equivalent to a projective line over a strong Jordan system of  $R$ .

Projective lines over strong Jordan systems admit many applications: For example, one may use them to describe subsets of Grassmannians which are closed

under reguli [23] or chain spaces on quadrics [4]. See also [3], [25], [26], [27], and the numerous examples given in [11].

Finally, let us mention one of the many questions that remain: *Is it possible to replace the strongness condition for Jordan systems by closedness under triple multiplication without affecting the known results?* A partial affirmative answer was given in [9] for case when  $R$  is the ring of  $n \times n$  matrices over a field  $F$  with an involution  $\sigma$  and  $J$  is the (not necessarily strong) Jordan system of  $\sigma$ -Hermitian matrices. The proof is based upon the verification that the projective line over this  $J$  is, up to some notational differences, nothing but the point set of a *dual polar space* [14] or, in the terminology of [38], the point set of a *projective space of  $\sigma$ -Hermitian matrices*.

A wealth of further references can be found in [2], [11], [19], [24], [28], [35], [37], and [38]. Refer to [12], [13], [17], [20], [21], [22], and [32] for deviating definitions of projective lines which we cannot present here.

## References

- [1] C. Bartolone. Jordan homomorphisms, chain geometries and the fundamental theorem. *Abh. Math. Sem. Univ. Hamburg*, 59:93–99, 1989.
- [2] W. Benz. *Vorlesungen über Geometrie der Algebren*. Springer, Berlin, 1973.
- [3] A. Blunck. Chain spaces over Jordan systems. *Abh. Math. Sem. Univ. Hamburg*, 64:33–49, 1994.
- [4] A. Blunck. Chain spaces via Clifford algebras. *Monatsh. Math.*, 123:98–107, 1997.
- [5] A. Blunck. *Geometries for Certain Linear Groups over Rings — Construction and Coordinatization*. Habilitationsschrift, Technische Universität Darmstadt, 1997.
- [6] A. Blunck. Projective groups over rings. *J. Algebra*, 249:266–290, 2002.
- [7] A. Blunck and H. Havlicek. The connected components of the projective line over a ring. *Adv. Geom.*, 1:107–117, 2001.
- [8] A. Blunck and H. Havlicek. Jordan homomorphisms and harmonic mappings. *Monatsh. Math.*, 139:111–127, 2003.
- [9] A. Blunck and H. Havlicek. Projective lines over Jordan systems and geometry of Hermitian matrices. *Linear Algebra Appl.*, 433:672–680, 2010.

- [10] A. Blunck and H. Havlicek. Geometric structures on finite- and infinite-dimensional Grassmannians. *Beitr. Algebra Geom.*, to appear.
- [11] A. Blunck and A. Herzer. *Kettengeometrien – Eine Einführung*. Shaker Verlag, Aachen, 2005.
- [12] U. Brehm. Algebraic representation of mappings between submodule lattices. *J. Math. Sci. (N. Y.)*, 153(4):454–480, 2008.
- [13] U. Brehm, M. Greferath, and S. E. Schmidt. Projective geometry on modular lattices. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 1115–1142. Elsevier, Amsterdam, 1995.
- [14] P. J. Cameron. Dual polar spaces. *Geom. Dedicata*, 12(1):75–85, 1982.
- [15] H. Chen. *Rings Related to Stable Range Conditions*, volume 11 of *Series in Algebra*. World Scientific, Singapore, 2011.
- [16] P. M. Cohn. On the structure of the  $GL_2$  of a ring. *Inst. Hautes Etudes Sci. Publ. Math.*, 30:365–413, 1966.
- [17] C.-A. Faure. Morphisms of projective spaces over rings. *Adv. Geom.*, 4(1):19–31, 2004.
- [18] D. Goldstein, R. M. Guralnick, L. Small, and E. Zelmanov. Inversion invariant additive subgroups of division rings. *Pacific J. Math.*, 227(2):287–294, 2006.
- [19] H. Havlicek. From pentacyclic coordinates to chain geometries, and back. *Mitt. Math. Ges. Hamburg*, 26:75–94, 2007.
- [20] H. Havlicek, A. Matraš, and M. Pankov. Geometry of free cyclic submodules over ternions. *Abh. Math. Semin. Univ. Hambg.*, 81(2):237–249, 2011.
- [21] H. Havlicek, J. Kosiorek, and B. Odehnal. A point model for the free cyclic submodules over ternions. *Results Math.*, to appear.
- [22] H. Havlicek and M. Saniga. Vectors, cyclic submodules, and projective spaces linked with ternions. *J. Geom.*, 92(1-2):79–90, 2009.
- [23] A. Herzer. On sets of subspaces closed under reguli. *Geom. Dedicata*, 41:89–99, 1992.
- [24] A. Herzer. Chain geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 781–842. Elsevier, Amsterdam, 1995.

- [25] A. Herzer. Konstruktion von Jordansystemen. *Mitt. Math. Ges. Hamburg*, 27:203–210, 2008.
- [26] A. Herzer. Die kleine projektive Gruppe zu einem Jordansystem. *Mitt. Math. Ges. Hamburg*, 29:157–168, 2010.
- [27] A. Herzer. Korrektur und Ergänzung zum Artikel *Die kleine projektive Gruppe zu einem Jordansystem* in *Mitt. Math. Ges. Hamburg* 29, Armin Herzer. *Mitt. Math. Ges. Hamburg*, 30:15–17, 2011.
- [28] L.-P. Huang. *Geometry of Matrices over Ring*. Science Press, Beijing, 2006.
- [29] H.-J. Kroll. Unterräume von Kettengeometrien und Kettengeometrien mit Quadrikenmodell. *Results Math.*, 19:327–334, 1991.
- [30] H.-J. Kroll. Unterräume von Kettengeometrien. In N. K. Stephanidis, editor, *Proceedings of the 3rd Congress of Geometry (Thessaloniki, 1991)*, pages 245–247, Thessaloniki, 1992. Aristotle Univ.
- [31] H.-J. Kroll. Zur Darstellung der Unterräume von Kettengeometrien. *Geom. Dedicata*, 43:59–64, 1992.
- [32] A. Lashkhi. Harmonic maps over rings. *Georgian Math. J.*, 4:41–64, 1997.
- [33] O. Loos. *Jordan Pairs*, volume 460 of *Lecture Notes in Mathematics*. Springer, Berlin, 1975.
- [34] S. Mattarei. Inverse-closed additive subgroups of fields. *Israel J. Math.*, 159:343–347, 2007.
- [35] M. Pankov. *Grassmannians of Classical Buildings*, volume 2 of *Algebra and Discrete Mathematics*. World Scientific, Singapore, 2010.
- [36] F. D. Veldkamp. Projective ring planes and their homomorphisms. In R. Kaya, P. Plaumann, and K. Strambach, editors, *Rings and Geometry*, pages 289–350. D. Reidel, Dordrecht, 1985.
- [37] F. D. Veldkamp. Geometry over rings. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 1033–1084. Elsevier, Amsterdam, 1995.
- [38] Z.-X. Wan. *Geometry of Matrices*. World Scientific, Singapore, 1996.